Math 2130
HW 4 Solutions

(1) $f(x,y) = 4x^2 - y^2 + 2y$

$$f_x(x,y) = 8x$$

$$f_y(x,y) = -2y + 2$$

(a) Note that

$$f(-1,2) = 4\cdot(-1)^2 - (2)^2 + 2(2) = 4$$

So, indeed, $P = (-1, 2, 4)$ does lie on the surface $z = f(x,y)$.

We have

$$f_x(-1,2) = 8(-1) = -8$$

$$f_y(-1,2) = -2(2) + 2 = -2$$

The tangent plane at $P = (-1, 2, 4)$ is

$$z - f(-1,2) = f_x(-1,2)\left[x - (-1)\right] + f_y(-1,2)\left[y - 2\right]$$

$$z - 4 = -8(x - (-1)) - 2(y - 2)$$

$$z-4=-8(x+1)-2(y-2)$$

$$\boxed{8x+2y+z=0}$$

(b) The tangent plane at $Q=(a,b,c)$ is

$$z-f(a,b)=f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

This is horizontal when $f_x(a,b)=0$ and $f_y(a,b)=0$, that is when the tangent plane equation is $z=f(a,b)$.

This is when

$$\boxed{\begin{array}{c} 8a=0 \\ -2b+2=0 \end{array}} \leftarrow \begin{array}{l} f_x(a,b)=0 \\ f_y(a,b)=0 \end{array}$$

This is when $a=0$ and $b=1$.

When $a=0$, $b=1$, we have

$$c = 4a^2 - b^2 + 2b = 4\cdot 0^2 - 1^2 + 2\cdot 1 = 1$$

$f(x,y) = 4x^2 - y^2 + 2y$

So, $Q=(0,1,1)$ is the only point where the tangent plane is horizontal.

2 $f(x,y) = y\ln(x)$

$f_x = y/x$

$f_y = \ln(x)$

(a) Note that

$$f(1,4) = 4\cdot\ln(1) = 4(0) = 0$$

So, indeed, $P=(1,4,0)$ does lie on the surface $z=f(x,y)$.

We have

$$f_x(1,4) = 4/1 = 4$$

$$f_y(1,4) = \ln(1) = 0$$

The tangent plane at $P=(1,4,0)$ is

$$z - f(1,4) = f_x(1,4)[x-1] + f_y(1,4)[y-4]$$

$$z - 0 = 4(x-1) - 0(y-4)$$

$$\boxed{-4x + z = -4}$$

$\boxed{\text{(b)}}$ The tangent plane at $Q=(a,b,c)$ is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This is horizontal when $f_x(a,b)=0$ and $f_y(a,b)=0$, that is when the tangent plane equation is $z=f(a,b)$.

This is when

$$\boxed{\begin{aligned} b/a &= 0 \\ \ln(a) &= 0 \end{aligned}}$$

$f_x(a,b)=0$
$f_y(a,b)=0$

This is when $a=1$ and $b=0$

When $a=1, b=0$, we have

$$c = 0\cdot\ln(1) = 0$$

$f(x,y)=y\ln(x)$

So, $Q = (1, 0, 0)$ is the only point where the tangent plane is horizontal.

3

$$f(x,y) = 9 - 2x + 4y - x^2 - 4y^2$$

Step 1: Find the critical points.

$$f_x = -2 - 2x$$
$$f_y = 4 - 8y$$

$$\boxed{\begin{aligned} -2-2x &= 0 \\ 4-8y &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} x &= -1 \\ y &= \tfrac{1}{2} \end{aligned}}$$

The only critical point is

$$(x,y) = \left(-1, \tfrac{1}{2}\right)$$

Step 2: Use the 2nd derivative test.

$$f_{xx} = -2 \qquad f_{xy} = 0$$
$$f_{yy} = -8$$

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$$= (-2)(-8) - 0^2 = 16 > 0$$

So at $(-1, \frac{1}{2})$ we have $D > 0$.

And $f_{xx}(-1, \frac{1}{2}) = -2 < 0$

Thus, $(-1, \frac{1}{2})$ is a local maximum.

4

$$f(x,y) = x^3 - 3x + y^3 - 3y$$

Step 1: Find the critical points.

$$f_x = 3x^2 - 3$$

$$f_y = 3y^2 - 3$$

$$\boxed{\begin{aligned} 3x^2 - 3 &= 0 \\ 3y^2 - 3 &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} x^2 - 1 &= 0 \\ y^2 - 1 &= 0 \end{aligned}} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

(1) gives $x = \pm 1$.

(2) gives $y = \pm 1$.

We need (1) and (2) to be solved simultaneously, so the critical points are

$$\boxed{(x,y) = (1,1), (1,-1), (-1,1), (-1,-1)}$$

Step 2: Use the 2nd derivative test.

$f_x = 3x^2 - 3$ $\quad$ $f_{xx} = 6x$ $\quad$ $f_{xy} = 0$

$f_y = 3y^2 - 3$ $\quad$ $f_{yy} = 6y$

$$D = f_{xx} f_{yy} - f_{xy}^2 = 36xy$$

Let's check each critical point.

$D(1,1) = 36(1)(1) = 36 > 0$

$f_{xx}(1,1) = 6(1) = 6 > 0$

So, $(1,1)$ is a local minimum.

$D(-1,-1) = 36(-1)(-1) = 36 > 0$

$f_{xx}(-1,-1) = 6(-1) = -6 < 0$

So, $(-1,-1)$ is a local maximum

$$\boxed{\begin{gathered} D(1,-1) = 36(1)(-1) = -36 < 0 \\ (1,-1) \text{ is a saddle point.} \end{gathered}}$$

$$\boxed{\begin{gathered} D(-1,1) = 36(-1)(1) = -36 < 0 \\ (-1,1) \text{ is a saddle point.} \end{gathered}}$$

5

$$f(x,y) = x^4 + y^4 - 4xy + 2$$

Step 1: Find the critical points.

$$f_x = 4x^3 - 4y$$
$$f_y = 4y^3 - 4x$$

$$\boxed{\begin{aligned} 4x^3 - 4y &= 0 \\ 4y^3 - 4x &= 0 \end{aligned}} \rightarrow \boxed{\begin{aligned} x^3 - y &= 0 \quad (1) \\ y^3 - x &= 0 \quad (2) \end{aligned}}$$

We want the (x,y) that solve both (1) and (2) at the same time.

(1) gives $y = x^3$.

Plug this into (2) to get:

$$y^3 - x = 0$$

$$(x^3)^3 - x = 0$$

$$x^9 - x = 0$$

$$x(x^8 - 1) = 0$$

$$x(x^4 - 1)(x^4 + 1) = 0$$

$$x\underbrace{(x^2 - 1)}\underbrace{(x^2 + 1)}\underbrace{(x^4 + 1)} = 0$$

$$\boxed{x = 0} \quad \boxed{\begin{array}{c} x^2 - 1 = 0 \\ x = \pm 1 \end{array}} \quad \boxed{\begin{array}{c} x^2 + 1 = 0 \\ x^2 = -1 \\ \text{can't} \\ \text{happen} \end{array}} \quad \boxed{\begin{array}{c} x^4 + 1 = 0 \\ x^4 = -1 \\ \text{can't} \\ \text{happen} \end{array}}$$

So, $x = 0, -1, 1$.

Plug these back into $y = x^3$ from 1 to get:

$x=0 \rightarrow y=0^3=0$

$x=-1 \rightarrow y=(-1)^3=1$

$x=1 \rightarrow y=1^3=1$

So, the critical points are:

$(x,y)=(0,0),\ (-1,1),\ (1,1).$

<u>Step 2:</u> Use the 2nd derivative test.

$f_x = 4x^3-4y$ $\quad f_{xx}=12x^2$

$f_y = 4y^3-4x$ $\quad f_{yy}=12y^2$

$f_{xy}=-4$

$$D=f_{xx}f_{yy}-f_{xy}^2=(12x^2)(12y^2)-(-4)^2$$

$$D=144x^2y^2-16$$

$$D(0,0) = 144(0)^2(0)^2 - 16 < 0$$

So, $(0,0)$ is a saddle point

$$D(1,1) = 144(1)^2(1)^2 - 16 = 128 > 0$$

$$f_{xx}(1,1) = 12(1)^2 = 12 > 0$$

So, $(1,1)$ is a local minimum

$$D(-1,-1) = 144(-1)^2(-1)^2 - 16 = 128 > 0$$

$$f_{xx}(-1,-1) = 12(-1)^2 = 12 > 0$$

So, $(-1,-1)$ is a local minimum

6 $f(x,y) = e^{-(x^2+y^2)} = e^{-x^2-y^2}$

Step 1: Find the critical points.

$f_x = -2x e^{-x^2-y^2}$

$f_y = -2y e^{-x^2-y^2}$

Need to solve

$$\boxed{\begin{aligned} -2x e^{-x^2-y^2} &= 0 \quad (1) \\ -2y e^{-x^2-y^2} &= 0 \quad (2) \end{aligned}}$$

Consider 1:

$$\underbrace{-2x}_{\{-2x=0\}} \underbrace{e^{-x^2-y^2}}_{\{e^{-x^2-y^2}=0\}} = 0$$

This is a product equalling 0.

Thus, either

$$-2x = 0 \quad \text{or} \quad e^{-x^2-y^2} = 0$$

But $e^{-x^2-y^2} > 0$ always.

So, we would need $-2x = 0$

That is, $x = 0$.

Similarly $-2y\underbrace{e^{-x^2-y^2}}_{>0} = 0$

would imply $-2y = 0$ or $y = 0$.

Thus, $x = 0, y = 0$.

So, there is one critical point at $(x,y) = (0,0)$.

Step 2: Use the 2nd derivative test.

$$f_x = -2xe^{-x^2-y^2}$$

$$f_y = -2ye^{-x^2-y^2}$$

$$f_{xx} = -2e^{-x^2-y^2} - 2x\left(-2xe^{-x^2-y^2}\right)$$

$$= (-2+4x^2)e^{-x^2-y^2}$$

$$f_{yy} = -2e^{-x^2-y^2} - 2y\left(-2ye^{-x^2-y^2}\right)$$

$$= (-2+4y^2)e^{-x^2-y^2}$$

$$f_{xy} = -2x\left(-2ye^{-x^2-y^2}\right)$$

$$= 4xye^{-x^2-y^2}$$

$$D(0,0) = f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}(0,0)^2$$

$$= \left[(-2+4\cdot 0^2)e^{-0^2-0^2}\right]\left[(-2+4\cdot 0^2)e^{-0^2-0^2}\right]$$

$$- \left[4\cdot 0\cdot 0\, e^{-0^2-0^2}\right]$$

$$= \left[-2e^0\right]\left[-2e^0\right] - \left[0\right]$$

$$= 4 > 0$$

$$f_{xx}(0,0) = \left[(-2+4\cdot 0^2)e^{-0^2-0^2}\right]$$

$$= \left[-2e^0\right] = -2 < 0$$

Thus, $(0,0)$ is a local maximum

7 $f(x,y) = \ln(1+x^2+y^2)$

Step 1: Find the critical points.

$$f_x = \frac{1}{1+x^2+y^2} \cdot (2x) = \frac{2x}{1+x^2+y^2}$$

$$f_y = \frac{1}{1+x^2+y^2}(2y) = \frac{2y}{1+x^2+y^2}$$

Need to solve

$$\frac{2x}{1+x^2+y^2} = 0 \quad (1)$$

$$\frac{2y}{1+x^2+y^2} = 0 \quad (2)$$

Recall that $\frac{a}{b} = 0$ if and only if $a = 0$.

Thus, ① and ② become

$$\boxed{\begin{aligned} 2x &= 0 \quad (1) \\ 2y &= 0 \quad (2) \end{aligned}} \longrightarrow \boxed{\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}}$$

$\boxed{\text{The only critical point is } (x, y) = (0, 0)}$

Step 2: Use the 2nd derivative test.

$$f_x = \frac{2x}{1 + x^2 + y^2} \qquad f_y = \frac{2y}{1 + x^2 + y^2}$$

$\boxed{\text{Quotient rule}}$ $\boxed{\left(\frac{a}{b}\right)' = \frac{a'b - ab'}{b^2}}$

$$f_{xx} = \frac{(2)(1+x^2+y^2) - 2x(2x)}{(1+x^2+y^2)^2}$$

$$= \frac{2 - 2x^2 + 2y^2}{(1+x^2+y^2)^2}$$

$$f_{yy} = \frac{2(1+x^2+y^2) - (2y)(2y)}{(1+x^2+y^2)^2}$$

$$= \frac{2 + 2x^2 - 2y^2}{(1+x^2+y^2)^2}$$

$$f_{xy} = \frac{0(1+x^2+y^2) - 2x(2y)}{(1+x^2+y^2)^2} = \frac{-4xy}{(1+x^2+y^2)^2}$$

$$D(0,0) = f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}(0,0)^2$$

$$= \frac{2 - 2\cdot 0^2 + 2\cdot 0^2}{(1+0^2+0^2)^2} \cdot \frac{2+2\cdot 0^2 - 2\cdot 0^2}{(1+0^2+0^2)^2}$$

$$- \frac{4\cdot 0\cdot 0}{(1+0^2+0^2)^2}$$

$$= \frac{2}{1}\cdot\frac{2}{1} - 0 = 4 > 0$$

$$f_{xx}(0,0) = \frac{2-2\cdot 0^2+2\cdot 0^2}{(1+0^2+0^2)^2} = 2 > 0$$

Thus, $(0,0)$ is a local minimum.